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On a variational inequality on elasto-hydrodynamic lubrication ☆

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ABSTRACT

We consider the problem of a deformable surface moving over a flat plane. The surfaces are separated by a small gap filled by a lubricant fluid. The mathematical model consists of the Reynolds variational inequality with nonlocal coefficients given by an integral operator which depends on the fluid pressure. The nonlocal operator represents the deformation of the lubricated surfaces. The problem considers the vertical displacement of the elastic surface from its reference configuration. The goal of the paper is to obtain the range of these admissible displacements. We present general results for nonlocal coefficients with applications to particular problems in elasto-hydrodynamic lubrication.

1. Introduction

Elasto-hydrodynamic problems in lubricated systems appear in many different industrial devices such as journal bearings, rolling contact bearings, gears, etc. The system consists of a coupling between a thin film flow and an elastic deformation of the surrounding surfaces (see for example [8]). Elasto-hydrodynamic problems in lubricated systems are a particular case of fluid-structure interaction.

The problem considers the displacement of the elastic surface from its reference configuration, which is unknown, as well as the pressure of the fluid. We assume that the pressure in the fluid obeys the Reynolds variational inequality (see [9]) frequently used to modelize the cavitation in thin fluid dynamics. We also assume that the elastic deformation of the upper surface is given by a positive operator τ, which depends on the fluid pressure; τ is in the field's literature a linear and integral operator, nevertheless we consider in this article a more general operator. The lower surface is considered rigid and known.

We consider the one-dimensional problem in $\Omega =]-1,1[$ and denote by K the positive cone in $H_0^1(\Omega)$, i.e.

$$K = \{\varphi \in H_0^1(\Omega),\ \varphi \geqslant 0\}.$$

The elasto-hydrodynamic problem consists of the following Reynolds variational inequality with nonlocal coefficients

$$\int_\Omega h^3(p)p' \cdot (\varphi - p)' \geqslant \int_\Omega h(p)(\varphi - p)', \quad \forall \varphi \in K, \tag{1.1}$$

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where p is the pressure of the fluid and $h(p) := h_1 + \tau(p)$, is the distance between the surfaces. In the above equality, $h_1(x) > 0$ is a given initial gap of the surfaces when no elastic deformation is considered and τ is a given positive operator from $H_0^1(\Omega)$ to a convenient space of functions on Ω.

Existence, uniqueness and regularity of the solution of this type of problems, with different operators τ has been studied in [1,3,11–13].

However, for most of practical engineering problems h_1 is not a given data. It is related to some geometrical degrees of freedom for the relative positions of the surrounding surfaces of the device. These degrees of freedom allow to the device to answer to some external forces acting on the device.

As a simplified model problem, we assume that the adimensional fluid gap without elastic deformation is given by

$$\left\{(x, y) \in \mathbb{R}^2,\ x \in \Omega,\ 0 \leqslant y \leqslant h_0(x) + \tilde{b}\right\}$$

where h_0 is a given reference form and $\tilde{b} > 0$ is a degree of freedom, which allows vertical translations of the upper surface. We assume that $h_0 \geqslant 0$ and $\min_{x\in\Omega} h_0(x) = 0$.

Then the effective fluid gap after deformation of the elastic surface is given by

$$h(x) = h_0(x) + \tilde{b} + \tau(p)(x). \tag{1.2}$$

The total balance of the forces is given by the equilibrium low

$$\int_\Omega p(x)\,dx = F \tag{1.3}$$

where F is the load applied on the device, assumed positive and known. Then the problem is to solve the coupled system (1.1)–(1.3) for the unknowns p and the displacement $\tilde{b}$.

The main question in problems of load is to know if there exists a solution of (1.1)–(1.3) for any given $F > 0$.

A natural approach to answer this question is to define a function $g :]0, +\infty[\to \mathbb{R}$ given by

$$g(\tilde{b}) = \int_\Omega p(x)\,dx$$

where for any $\tilde{b} > 0$, $p = p(\tilde{b})$ is the solution of (1.1) and (1.2). This approach has been used in different fluid-rigid structure interaction problems in the field (see for instance [4–7]).

It is not difficult to prove that g is well defined, continuous and satisfies

$$\lim_{\tilde{b}\to+\infty} g(\tilde{b}) = 0.$$

On the other hand, we can easily prove that g can be extended by continuity to $\tilde{b} = 0$; let us denote $F_0 = g(0) > 0$. Then, the solution $(p, \tilde{b})$ to the problem exists for any F between 0 and F_0.

In order to obtain a maximum range of admissible forces F we have to extend the function g to negative values, that is, to allow virtual negative translations $\tilde{b}$. The restriction on the domain of definition of g must be in such manner that the distance h, after deformation, must be positive.

Then it is desirable to extend the function g to an interval $]\tilde{b}_0, +\infty[$ with $-\infty \leqslant \tilde{b}_0 < 0$ such that

$$\lim_{\tilde{b}\to\tilde{b}_0} g(\tilde{b}) = +\infty.$$

A complete answer to this question is a difficult task. In this paper we present only partial results; we give intervals of negative values of $\tilde{b}$ for which the existence of a solution p to (1.1) and (1.2) is assured.

Throughout this paper we consider a fixed constant $b \geqslant 0$ and we treat with the problem

$$\begin{cases} \text{find } p \in K \text{ such that} \\ \displaystyle\int_\Omega h^3(p)p' \cdot (\varphi - p)' \geqslant \int_\Omega h(p)(\varphi - p)', \quad \forall \varphi \in K, \\ h(p)(x) > 0, \quad \forall x \in \Omega, \end{cases} \tag{1.4}$$

where h is given by

$$h(p)(x) = h_0(x) + \tau(p)(x) - b. \tag{1.5}$$

We remark that the main difficulty of the problem is to prevent non-positive values of the distance $h(p)$, which might occur due to the presence of the term "$-b$". This is the main difference with the previous works on the subject (see [1,3,11–13]) where $b < 0$ and therefore $h(p)$ is assured to take only positive values.

The first result presented concerns the local existence: we prove that under weak enough assumptions on h_0 and on the operator τ, there exists $b_0 > 0$ such that for any $b \in [0, b_0[$ a solution to (1.4)–(1.5) exists.

The second result presented is more "global" in nature: under stronger assumptions on h_0 and τ, we give a constant value $b_0 > 0$ depending only on the data h_0 such that the solution to (1.4)–(1.5) exists for any $b \in [0, b_0[$. Both results are stated in a single theorem in Section 2.

We also present some examples and applications of the theoretical results with practical importance. First example considers the integral operator expressing the Hertz low contact and we verify that it satisfies the required weak assumptions. Second and third examples are applications to the Euler–Bernoulli beam equation and the "regularized Hertz operator" respectively. These operators satisfy the strong assumptions of the theorem.

In Section 2 we set the hypothesis as well as the main results of this paper. Section 3 is devoted to the proof of the main result and in Section 4 we present some examples of the theoretical results.

2. Hypothesis and main result

Let us denote $\Omega_\delta =]-1+\delta, 1-\delta[$ for any $\delta \in]0, \frac{1}{2}[$. For any continuous and positive function $w : \Omega \to]0, +\infty[$ we set

$$L^1_w(\Omega) = L^1(\Omega, w\,dx) = \left\{u : \Omega \to \mathbb{R},\ u \text{ measurable},\ \int_\Omega u(x)w(x)\,dx < +\infty\right\}.$$

Notice that if $w \in L^2(\Omega)$ then $L^2(\Omega) \subset L^1_w(\Omega)$.

We have the following general hypothesis on h_0:

$$h_0 \in C^1(\bar{\Omega}), \tag{2.1}$$

$$\min_{x\in[-1,1]} h_0(x) = h_0(0) = 0 < \min\{h_0(-1), h_0(1)\}. \tag{2.2}$$

The general hypothesis on τ is the following:

There exists a continuous and positive weight $w : \Omega \to]0, +\infty[$ with $w \in L^2(\Omega)$ such that

$$\tau : L^1_w(\Omega) \to L^6_{loc}(\Omega)$$

and satisfies

$$\tau \text{ is continuous in 0 in the sense that for any } \delta > 0,$$

$$\varphi_n \to 0 \quad \text{in } L^1_w(\Omega) \quad \text{implies} \quad \tau(\varphi_n) \to 0 \quad \text{in } L^6(\Omega_\delta), \tag{2.3}$$

$$\tau \text{ is continuous from } L^2(\Omega) \text{ to } C(\bar{\Omega}), \tag{2.4}$$

$$\tau(\varphi) \geqslant 0 \quad \text{for any } \varphi \geqslant 0, \tag{2.5}$$

$$\text{for any } \delta \in \left]0, \frac{1}{2}\right[\text{ there exists } c(\delta) > 0, \text{ such that}$$

$$\inf_{x\in\Omega_\delta} \tau(\varphi) \geqslant c(\delta) \int_\Omega \varphi(x)w(x)\,dx, \quad \forall \varphi \in L^2(\Omega),\ \varphi \geqslant 0. \tag{2.6}$$

The main result of this paper is presented in the following theorem:

Theorem 1. *We assume h_0 satisfies the general assumptions* (2.1), (2.2) *and τ fulfills* (2.3)–(2.6)*. Then*:

(i) *There exists $b_0 > 0$ such that for any $b \in [0, b_0[$ there exists at least a solution to* (1.4) *and* (1.5)*.*

(ii) *Assume that the following supplementary hypotheses on h_0 and τ are satisfied*

$$h_0 \in W^{3,d}_{loc}(\Omega), \tag{2.7}$$

$$\textit{for any } \delta \in \left]0, \frac{1}{2}\right[\textit{ and any } B \subset L^1_w(\Omega) \textit{ bounded},\ \tau(B) \textit{ is bounded in } W^{3,d}(\Omega_\delta). \tag{2.8}$$

Then for any b satisfying

$$0 \leqslant b < \min\{h_0(-1), h_0(1)\}, \tag{2.9}$$

there exists at least a solution to (1.4) *and* (1.5)*.*

3. Proof of Theorem 1

In order to prove the theorem we first introduce the following technical lemma.

Lemma 1. *Let $I_\epsilon =]\eta_1^\epsilon, \eta_2^\epsilon[\subset \mathbb{R}$ an open and bounded interval and let $A_\epsilon \in L^\infty(I_\epsilon)$. We assume that there exist positive constants $\epsilon_0, c_1, c_2, c_3$ such that for any $\epsilon \in]0, \epsilon_0]$ we have*

$$c_1\epsilon \leqslant A_\epsilon(x) \leqslant c_2\epsilon \quad \text{a.e. } x \in I_\epsilon \tag{3.1}$$

and

$$-A'_\epsilon \geqslant c_3\epsilon \quad \text{in } \mathcal{D}'(I_\epsilon). \tag{3.2}$$

Let z_ϵ be the solution to the problem

$$\left(A_\epsilon^3 z'_\epsilon\right)' = A'_\epsilon \quad \text{in } I_\epsilon, \tag{3.3}$$

$$z_\epsilon = 0 \quad \text{in } \partial I_\epsilon \tag{3.4}$$

then, there exists $c_4 > 0$ such that

$$\int_{I_\epsilon} z_\epsilon \geqslant c_4 \frac{|\eta_2^\epsilon - \eta_1^\epsilon|^3}{\epsilon^2}, \quad \forall \epsilon \in]0, \epsilon_0]. \tag{3.5}$$

Proof. We consider the problem

$$-\left(A_\epsilon^3 \hat{z}'_\epsilon\right)' = c_3\epsilon \quad \text{in } I_\epsilon, \tag{3.6}$$

$$\hat{z}_\epsilon = 0 \quad \text{in } \partial I_\epsilon, \tag{3.7}$$

and we deduce, by maximum principle and assumption (3.2),

$$z_\epsilon \geqslant \hat{z}_\epsilon. \tag{3.8}$$

The weak formulation of (3.6), (3.7) is given by

$$\int_{I_\epsilon} A_\epsilon^3 \hat{z}'_\epsilon \varphi' = c_3\epsilon \int_{I_\epsilon} \varphi \quad \text{for any } \varphi \in H_0^1(I_\epsilon). \tag{3.9}$$

Taking $\varphi = \hat{z}_\epsilon$, we obtain

$$\int_{I_\epsilon} A_\epsilon^3 \left|\hat{z}'_\epsilon\right|^2 = c_3\epsilon \int_{I_\epsilon} \hat{z}_\epsilon. \tag{3.10}$$

From (3.9) and Holder inequality we have

$$\int_{I_\epsilon} A_\epsilon^3 \left|\hat{z}'_\epsilon\right|^2 \geqslant c_3^2\epsilon^2 \frac{(\int_{I_\epsilon} \varphi)^2}{\int_{I_\epsilon} A_\epsilon^3 |\varphi'|^2} \quad \text{for any } \varphi \in H_0^1(\Omega). \tag{3.11}$$

We take φ in (3.11) defined by $\varphi = \xi(\frac{x-\eta_1^\epsilon}{\eta_2^\epsilon - \eta_1^\epsilon})$ where $\xi \in H_0^1(]0,1[)$ is independent of ϵ and satisfies $\int_0^1 \xi > 0$. Since

$$\int_{I_\epsilon} \varphi = \left(\eta_2^\epsilon - \eta_1^\epsilon\right) \int_0^1 \xi$$

and

$$\int_{I_\epsilon} A_\epsilon^3 \left|\varphi'\right|^2 dx \leqslant \frac{c_2^3\epsilon^3}{|\eta_2^\epsilon - \eta_1^\epsilon|} \int_0^1 \left|\xi'\right|^2$$

we obtain from (3.11)

$$\int_{I_\epsilon} A_\epsilon^3 \left|\hat{z}_\epsilon'\right|^2 \geqslant c\frac{|\eta_2^\epsilon - \eta_1^\epsilon|^3}{\epsilon},$$

for a positive constant c independent of ϵ. Finally, from (3.10) and (3.8) we have the result. □

Proof of Theorem 1. Let ϵ be a positive number, we consider the problem

$$\begin{cases} \text{find } p_\epsilon \in K \text{ such that} \\ \int_\Omega h_\epsilon^3(p_\epsilon)p_\epsilon' \cdot (\varphi - p_\epsilon)' \geqslant \int_\Omega h_\epsilon(p_\epsilon)(\varphi - p_\epsilon)', \quad \forall \varphi \in K \end{cases} \tag{3.12}$$

where, for any $\varphi \in L_w^1(\Omega)$, $h_\epsilon(\varphi)$ is defined by

$$h_\epsilon(\varphi) = \max\{h(\varphi), \epsilon(a - x_1)\},$$

with $a > 1$ a fixed number.

Remark 1. Notice that $h_\epsilon(\varphi) \geqslant \epsilon(a-1) > 0$, for any $\varphi \in L_w^1(\Omega)$.

We split the proof of the theorem into several steps:

Step 1. Existence of solutions to (3.12).
We introduce the set S

$$S := \{\varphi \in L^2(\Omega),\ \varphi \geqslant 0\},$$

and the operator T_ϵ

$$T_\epsilon : S \to L^2(\Omega),$$

such that $T_\epsilon(p) = q_\epsilon$ where $q_\epsilon \in K$ is the unique solution to

$$\int_\Omega h_\epsilon^3(p)q_\epsilon'(\varphi - q_\epsilon)' \geqslant \int_\Omega h_\epsilon(p)(\varphi - q_\epsilon)', \quad \forall \varphi \in K. \tag{3.13}$$

In order to apply Schauder fixed point theorem, we verify that T_ϵ satisfies the assumptions of the theorem:

- T_ϵ is well defined and $T_\epsilon(S) \subset S$.
  By assumptions (2.1) and (2.4) we have that
  $$h_\epsilon(p) \in C(\bar{\Omega}) \quad \text{for all } p \in S.$$
  We also have from Remark 1 that $h_\epsilon(p) \geqslant \epsilon(a-1)$ for all $p \in S$.
  Then, (3.13) admits a unique classical solution $q_\epsilon \in K$ (see for instance Kinderlehrer and Stampacchia [10, Theorem 6.2, p. 40]).
- T_ϵ is continuous.
  It is a consequence of continuity (2.4) of τ and continuous dependence of solutions of variational inequalities with respect to data.
- T_ϵ is compact.
  We take $\varphi = 0$ as test function in (3.13) to obtain
  $$\int_\Omega h_\epsilon^3(p)|\nabla q_\epsilon|^2 \leqslant \int_\Omega h_\epsilon(p)\frac{\partial q_\epsilon}{\partial x}.$$
  Since
  $$\int_\Omega h_\epsilon(p)\frac{\partial q_\epsilon}{\partial x} \leqslant \left[\int_\Omega h_\epsilon^3(p)|\nabla q_\epsilon|^2\right]^{\frac{1}{2}}\left[\int_\Omega \frac{1}{h_\epsilon(p)}\right]^{\frac{1}{2}}$$
  we deduce
  $$\int_\Omega h_\epsilon^3(p)|\nabla q_\epsilon|^2 \leqslant \int_\Omega \frac{1}{h_\epsilon(p)}. \tag{3.14}$$

Using the inequality $h_\epsilon(p) \geqslant \epsilon(a-1)$ we deduce

$$\int_\Omega |\nabla q_\epsilon|^2 \leqslant \frac{|\Omega|}{\epsilon^4(a-1)^4}.$$

The compactness of the operator T_ϵ follows from the Poincare inequality and the compactness of the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$.

By Schauder fixed point theorem we conclude the existence of at least one solution $p_\epsilon \in K$ to the problem [3.12].

We remark that the result given in Step 1 is valid for any $b \geqslant 0$ and is common to both parts of Theorem 1.

Step 2. In this step we only use the general assumptions (2.1)–(2.6) to prove the following statement: $\exists b_0 > 0$ and $\exists \epsilon > 0$ such that $\forall b \in [0, b_0[$, we have $h(p_\epsilon)(x) \geqslant \epsilon(a-x)$, $\forall x \in \Omega$. Clearly, this will prove the part (i) of Theorem 1, since $h_\epsilon(p_\epsilon) = h(p_\epsilon)$, so p_ϵ is a solution to (1.4).

By contradiction, we assume that the above assertion is false. Then, for any $\epsilon > 0$ there exist $b_\epsilon \in [0, \epsilon[$ and $x_\epsilon \in \Omega$ such that

$$\tau(p_\epsilon)(x_\epsilon) + h_0(x_\epsilon) < b_\epsilon + \epsilon(a - x_\epsilon). \tag{3.15}$$

Since $h_0(-1) > 0$, $h_0(1) > 0$ and $\tau(p_\epsilon) \geqslant 0$ we deduce from the continuity of h_0 that there exists $\gamma_1 > 0$ such that $x_\epsilon \in \Omega_{\gamma_1}$ if ϵ is small enough. With the help of the hypothesis (2.6) we deduce from (3.15) that

$$\int_\Omega p_\epsilon(x) w(x)\, dx \to 0 \quad \text{for } \epsilon \to 0. \tag{3.16}$$

On the other hand, since $h_0 \in C^1(\bar{\Omega})$, $h_0(-1) > 0$ and $h_0(0) = 0$, we deduce that there exists an interval I with $\bar{I} \subset \Omega$ and a number $h_I > 0$ such that $h_0' < 0$ and $h_0 \geqslant h_I$ on I. Let $\xi \in H_0^1(I)$ be the unique solution of the linear elliptic equation on I:

$$\left(h_0^3 \xi'\right)' = h_0'.$$

By the strong maximum principle we deduce that

$$\xi > 0 \quad \text{on } I. \tag{3.17}$$

Let us now consider the following problem for any $\epsilon > 0$ small enough

$$\begin{aligned} &\left[\left[h_0 + \tau(p_\epsilon) - b_\epsilon\right]^3 \xi_\epsilon'\right]' = \left[h_0 + \tau(p_\epsilon)\right]', \quad x \in I, \\ &\xi_\epsilon = 0, \quad x \in \partial I. \end{aligned}$$

The above problem is well posed since

$$h_0 + \tau(p_\epsilon) - b_\epsilon \geqslant \frac{h_I}{2} \quad \text{on } I \tag{3.18}$$

and $\tau(p_\epsilon) \in C(\bar{\Omega})$ (see (2.4)).

The variational formulation of the problem satisfied by ξ_ϵ is

$$\int_I \left[h_0 + \tau(p_\epsilon) - b_\epsilon\right]^3 \xi_\epsilon' \varphi' = \int_I \left[h_0 + \tau(p_\epsilon)\right]\varphi', \quad \forall \varphi \in H_0^1(I). \tag{3.19}$$

Taking $\varphi = \xi_\epsilon$ in the above relation and using (3.18) we easily deduce that ξ_ϵ is bounded uniformly in ϵ in the norm of $H_0^1(I)$. We deduce that there exists $\bar{\xi} \in H_0^1(I)$ such that we have, up to a subsequence in ϵ:

$$\xi_\epsilon \to \bar{\xi} \quad \text{in } H_0^1(I) \text{ weak and in } L^2(I) \text{ strong}.$$

Now from (3.16) and the hypothesis (2.3) we deduce that $\tau(p_\epsilon) \to 0$ in $L^6(I)$ strongly. Then we can pass to the limit as $\epsilon \to 0$ in (3.19) to obtain that for any $\varphi \in \mathcal{D}(I)$:

$$\int_I h_0^3 \bar{\xi}' \varphi' = \int_I h_0 \varphi'.$$

By density we deduce that the above equality is valid for any $\varphi \in H_0^1(I)$, which implies by uniqueness $\bar{\xi} = \xi$. Also remark that, by maximum principle we have

$$p_\epsilon \geqslant \xi_\epsilon \quad \text{on } I.$$

Since $p_\epsilon \geqslant 0$ on Ω, we deduce from the strong convergence in $L^2(I)$ of p_ϵ to ξ:

$$\int_\Omega p_\epsilon w \geqslant \frac{1}{2} \int_I \xi w$$

and this is in contradiction with (3.16) and (3.17).

Step 3. In this step we assume the supplementary assumptions (2.7) and (2.8) to prove that for any fixed b such that $0 \leqslant b < \min\{h_0(-1), h_0(1)\}$, there exists $\epsilon > 0$ such that $h(p_\epsilon)(x) \geqslant \epsilon(a - x)$, $\forall x \in \Omega$. This proves Theorem 1, part (ii).

Arguing by contradiction, we assume that for all $\epsilon > 0$ there exists a nonempty open set $\omega_\epsilon \subset \Omega$, such that $h(p_\epsilon) < \epsilon(a - x)$ for all $x \in \omega_\epsilon$ because the continuity of $h(p_\epsilon)$. For the sake of notation we denote by $\hat{H}_\epsilon$ the function $h_\epsilon(p_\epsilon)$ and by H_ϵ the function $h(p_\epsilon)$. Then we have

$$\hat{H}_\epsilon = \epsilon(a - x) \quad \text{in } \omega_\epsilon.$$

Notice that, from the continuity of h_0, the non-negativity of $\tau(p_\epsilon)$ and assumption (2.9), there exists $\gamma_2 > 0$ such that

$$\omega_\epsilon \subset \Omega_{\gamma_2} \quad \text{for } \epsilon \text{ small enough.} \tag{3.20}$$

Consider $x_0 \in \omega_\epsilon$, then $h_0(x_0) + \tau(p_\epsilon)(x_0) - b < \epsilon(a - x_0) \leqslant 1$ for ϵ small enough and we deduce

$$\tau(p_\epsilon)(x_0) < b + 1$$

(2.6) implies

$$\int_\Omega p_\epsilon(x) w(x)\, dx \leqslant \frac{b+1}{c(\gamma_2)} \quad \text{for } \epsilon \text{ small enough.} \tag{3.21}$$

Notice that ω_ϵ can be a non-connected set, and let us denote $]\alpha_\epsilon, \beta_\epsilon[$ its connected component closest to 0.

Let us fix $r \in]\frac{d}{2d-1}, \frac{2}{3}[$.

Case I. There exists ϵ_j subsequence of ϵ and $\kappa > 0$ independent of ϵ_j such that

$$\frac{\beta_{\epsilon_j} - \alpha_{\epsilon_j}}{\epsilon_j^r} \geqslant \kappa, \tag{3.22}$$

therefore, the sequence $(\beta_\epsilon - \alpha_\epsilon)/\epsilon^r$ does not converge to 0. For simplicity in the notation we drop the index "j". We now consider the following equation:

$$\begin{cases} \left(\hat{H}_\epsilon^3 \varphi_{1\epsilon}'\right)' = \hat{H}_\epsilon' & \text{in }]\alpha_\epsilon, \beta_\epsilon[, \\ \varphi_{1\epsilon}(\alpha_\epsilon) = \varphi_{1\epsilon}(\beta_\epsilon) = 0. \end{cases} \tag{3.23}$$

By maximum principle we have $p_\epsilon \geqslant \varphi_{1\epsilon}$ in $]\alpha_\epsilon, \beta_\epsilon[$.

Since $\hat{H}_\epsilon$ satisfies the assumptions (3.1) and (3.2) we can apply Lemma 1 with $A_\epsilon = \hat{H}_\epsilon$ and $I_\epsilon =]\alpha_\epsilon, \beta_\epsilon[$. Then

$$\int_{\alpha_\epsilon}^{\beta\epsilon} \varphi_{1\epsilon} \geqslant c_4 \frac{(\beta_\epsilon - \alpha_\epsilon)^3}{\epsilon^2}$$

and from (3.22) it results

$$\int_{\alpha_\epsilon}^{\beta\epsilon} \varphi_{1\epsilon} \geqslant c_4 \kappa^3 \epsilon^{3r-2}. \tag{3.24}$$

Since $r < \frac{2}{3}$ we have

$$\int_{\alpha_\epsilon}^{\beta\epsilon} \varphi_{1\epsilon} \to \infty \quad \text{so} \int_{\alpha_\epsilon}^{\beta\epsilon} p_\epsilon \to \infty$$

which contradicts (3.21).

Case II. $\frac{\beta_\epsilon - \alpha_\epsilon}{\epsilon^r} \to 0$.

Consider the interval $J_\epsilon :=]\alpha_\epsilon - \epsilon^r, \alpha_\epsilon[$ which is included in $\Omega_{\gamma_0/2}$ for ϵ small enough, due to (3.20). Now consider the problem

$$\begin{cases} \left(\hat{H}_\epsilon^3 \varphi'_{2\epsilon}\right)' = \hat{H}'_\epsilon & \text{in } J_\epsilon, \\ \varphi_{2\epsilon}\left(\alpha_\epsilon - \epsilon^r\right) = \varphi_{2\epsilon}(\alpha_\epsilon) = 0. \end{cases} \tag{3.25}$$

As in Case I we have that $p_\epsilon \geqslant \varphi_{2\epsilon}$ on J_ϵ, and the goal is now to prove

$$\int_{J_\epsilon} \varphi_{2\epsilon} \to +\infty \quad \text{when } \epsilon \to 0 \tag{3.26}$$

to contradicts (3.21). For this we will prove that on the interval J_ϵ the functions $\hat{H}_\epsilon$ and $-\hat{H}'_\epsilon$ behave as ϵ (see inequalities (3.29) and (3.30) later) and this gives the result using Lemma 1.

Let us set

$$g_\epsilon = H_\epsilon(x) - \epsilon(a - x).$$

Then

$$g_\epsilon(\alpha_\epsilon) = g(\beta_\epsilon) = 0,$$
$$g_\epsilon(x) < 0 \quad \text{in }]\alpha_\epsilon, \beta_\epsilon[.$$

Let $\gamma_\epsilon \in]\alpha_\epsilon, \beta_\epsilon[$ such that

$$\min_{x \in [\alpha_\epsilon, \beta_\epsilon]} g_\epsilon(x) = g_\epsilon(\gamma_\epsilon).$$

We deduce immediately that $H'_\epsilon(\gamma_\epsilon) = -\epsilon$ and $H''_\epsilon(\gamma_\epsilon) \geqslant 0$.

We now have from Taylor development

$$H'_\epsilon(x) = H'_\epsilon(\gamma_\epsilon) + H''_\epsilon(\gamma_\epsilon)(x - \gamma_\epsilon) + \int_{\gamma_\epsilon}^{x} (x - y) H'''_\epsilon(y)\, dy. \tag{3.27}$$

By Holder inequality we have for any $x \in J_\epsilon$

$$\left| \int_{\gamma_\epsilon}^{x} (x - y) H'''_\epsilon(y)\, dy \right| \leqslant \left\| H'''_\epsilon \right\|_{L^d(\Omega)} \left[\int_{\alpha_\epsilon - \epsilon^r}^{\gamma_\epsilon} \left(\gamma_\epsilon - \alpha_\epsilon + \epsilon^r\right)^{d'} \right]^{\frac{1}{d'}}$$

where $\frac{1}{d} + \frac{1}{d'} = 1$. Now for ϵ small enough we have

$$\gamma_\epsilon - \alpha_\epsilon + \epsilon^r \leqslant 2\epsilon^r.$$

From (3.21) and the supplementary hypotheses (2.7) and (2.8), we deduce

$$\|H_\epsilon\|_{W^{3,d}(\Omega_{\gamma_2/2})} \leqslant c_6$$

with c_6 independent of ϵ. It results

$$\left| \int_{\gamma_\epsilon}^{x} (x - y) H'''_\epsilon(y)\, dy \right| \leqslant c_7 \epsilon^{r(2 - 1/d)} \tag{3.28}$$

with c_7 independent of ϵ. Since $H'_\epsilon(\gamma_\epsilon) = -\epsilon$ and $H''_\epsilon(\gamma_\epsilon) \geqslant 0$ we deduce from (3.27) and (3.28),

$$-H'_\epsilon(x) \geqslant \epsilon - c_7 \epsilon^{r(2 - 1/d)}.$$

It is clear from the choice of r that

$$-H'_\epsilon \geqslant \frac{\epsilon}{2} \quad \text{on } J_\epsilon \text{ for } \epsilon \text{ small enough.} \tag{3.29}$$

On the other hand we have for any $x \in J_\epsilon$

$$H_\epsilon(x) = H_\epsilon(\gamma_\epsilon) + H'_\epsilon(\gamma_\epsilon)(x-\gamma_\epsilon) + \frac{1}{2}H''_\epsilon(\gamma_\epsilon)(x-\gamma_\epsilon)^2 \leqslant \epsilon(a+1) + \epsilon\big(2\epsilon^r\big) + \frac{c_8}{2}\big(2\epsilon^r\big)^2,$$

where we used the Sobolev embedding $W^{3,d}(\Omega) \hookrightarrow C^2(\bar{\Omega})$. From the choice of r we deduce

$$H_\epsilon \leqslant (a+2)\epsilon \quad \text{on } J_\epsilon \text{ for } \epsilon \text{ small enough.}$$

Taking into account that

$$\hat{H}_\epsilon(x) = \max\big\{H_\epsilon, \epsilon(a-x)\big\}$$

we have

$$(a-1)\epsilon \leqslant \hat{H}_\epsilon(x) \leqslant (a+2)\epsilon \quad \text{on } J_\epsilon. \tag{3.30}$$

We apply Lemma 1 with $I_\epsilon = J_\epsilon$ and $A_\epsilon = \hat{H}_\epsilon$, and with the help of (3.29) and (3.30) we obtain that

$$\int_{J_\epsilon} \varphi_{2\epsilon} \geqslant c_4\epsilon^{3r-2}.$$

From the choice of r we deduce (3.26), and the proof ends. □

4. Examples

In this section we present three different examples of applications of Theorem 1.

4.1. Example 1

A first example of operator τ that we present is known as the **Hertz elastic operator** and it is defined by

$$\tau_1(\varphi)(x) = \int_\Omega \log\left(\frac{M}{|x-y|}\right)\varphi(y)\,dy$$

where M is a positive constant bigger than 2 (see for instance Bayada, Martin, Vázquez [2]). Notice that τ_1 is linear and $\tau(\varphi)$ is the convolution between the functions $\{x \to \log(\frac{M}{|x|})\}$ and φ extended by 0 outside Ω.

It is easy to see that the general hypothesis (2.3)–(2.6) is satisfied with weight $w \equiv 1$. So the local result given in part (i) of Theorem 1 is valid for $\tau = \tau_1$.

On the other hand, the hypothesis (2.8) is not satisfied for this operator, so the existence of a solution to the problem (1.4) for $\tau = \tau_1$ and for any

$$b \in \big[0, \min\big\{h_0(-1), h_0(1)\big\}\big[$$

remains an open question.

4.2. Example 2

We consider here that the deformation of the upper surface is modeled by the **Euler–Bernoulli beam** equation:

$$\tau_2(p) = u, \quad \text{for } u \text{ satisfying } EI\frac{d^4}{dx^4}u = p, \text{ in }]-1,1[\tag{4.1}$$

where u is the beam deflection, p represents the pressure of the lubricant, E is the elastic modulus and I is the second moment of area. For simplicity we assume $EI = 1$.

We consider hinged boundary conditions (see Bayada, Cid and Vázquez [1] for details)

$$u(-1) = u(1) = 0, \tag{4.2}$$

$$u''(-1) = u''(1) = 0. \tag{4.3}$$

In Bayada, Cid and Vázquez [1], the authors consider the problem (1.4) for $b < 0$ and study the existence of solutions. Further analysis is studied for a nonlinear elastic model.

Now the variational formulation of the problem (4.1), (4.2) and (4.3) is: find $u \in H^2(\Omega) \cap H^1_0(\Omega)$ such that

$$\int_\Omega u''(x)\psi''(x)\,dx = \langle p, \psi\rangle, \quad \forall\psi \in H^2(\Omega) \cap H^1_0(\Omega), \tag{4.4}$$

with $p \in (H^2(\Omega) \cap H^1_0(\Omega))'$ given, where $\langle\,,\rangle$ is the duality product in $H^2(\Omega) \cap H^1_0(\Omega)$.

We have classically, by the Lax–Milgram theorem, the existence and uniqueness of a solution to (4.4). Then the operator τ_2 is a priori well defined as a linear and continuous operator from $L^1(\Omega)$ to $H^2(\Omega) \cap H_0^1(\Omega)$. Moreover, we can write τ_2 as an integral operator defined by the kernel $k_2 : \Omega^2 \to \mathbb{R}$ which satisfies

$$\frac{\partial^4}{\partial x^4} k_2(x, y) = \delta_y \quad \text{(Dirac distribution in } y\text{)}$$

with boundary conditions:

$$k_2(-1, y) = k_2(1, y) = \frac{\partial^2}{\partial x^2} k_2(-1, y) = \frac{\partial^2}{\partial x^2} k_2(-1, y) = 0.$$

It is easy to see that

$$k_2(x, y) := \frac{1}{12} \begin{cases} (y+1)(x-1)^3 + (y^2-1)(y+3)(x-1) & \text{if } y < x, \\ (y-1)(x+1)^3 + (y^2-1)(y-3)(x+1) & \text{if } y > x. \end{cases}$$

Then the operator is given by the formula

$$\tau_2(p)(x) = \int_{-1}^{1} k_2(x, y) p(y)\, dy, \quad \forall x \in \Omega. \tag{4.5}$$

We have the following result:

Proposition 1. *The hypotheses* (2.3)–(2.6) *and* (2.8) *are satisfied for the operator* $\tau = \tau_2$ *with weight* $w(x) = 1 - x^2$.

Proof. The assumption (2.4) is an obvious consequence of (4.4) or (4.5). We now fix $\delta \in]0, 1/2[$, then for any $x \in \Omega_\delta$ we have that $\frac{k_2(x,y)}{1-y^2}$ is bounded, as well as the expressions $\frac{\partial^j}{\partial x^j}(\frac{k_2(x,y)}{1-y^2})$, for $j = 1, 2, 3$. We deduce that τ_2 is well defined as a linear operator from $L_w^1(\Omega)$ to $W_{loc}^{3,d}(\Omega)$ and also that (2.3) and (2.8) are satisfied.

Let us introduce the function $\hat{k}_2 : \Omega^2 \to \mathbb{R}$ given by

$$\hat{k}_2(x, y) := \frac{1}{6} \begin{cases} (1+y)(1-x)^2(1+x) & \text{if } y < x, \\ (1-y)(1+x)^2(1-x) & \text{if } y > x. \end{cases}$$

Using that

$$(y-1)(y+3)(x-1) \geqslant (x-1)^2(x+3) \quad \text{for } -1 \leqslant y < x \leqslant 1$$

and

$$(y+1)(y-3)(x+1) \geqslant (x+1)^2(x-3) \quad \text{for } -1 \leqslant x < y \leqslant 1$$

we deduce

$$k_2(x, y) \geqslant \hat{k}_2(x, y), \quad \forall x, y \in \Omega. \tag{4.6}$$

Since $\hat{k}_2 \geqslant 0$, we obtain (2.5). We remark that (2.5) can also be obtained by using the maximum principle, which is valid for a fourth order equation with boundary conditions given by (4.2) and (4.3).

Let us consider $x \in \Omega_\delta$, with $\delta \in]0, 1/2[$. Using the inequalities

$$1 + y \geqslant \frac{1-y^2}{2} \quad \text{for } y < x$$

and

$$1 - y \geqslant \frac{1-y^2}{2} \quad \text{for } y > x$$

we deduce

$$\hat{k}_2(x, y) \geqslant \frac{\delta^3}{12}\left(1 - y^2\right), \quad \forall x \in \Omega_\delta,\ \forall y \in \Omega. \tag{4.7}$$

Then using (4.6) and (4.7) we obtain for any $\varphi \in L_w^1(\Omega)$, $\varphi \geqslant 0$:

$$\inf_{x\in\Omega_\delta} \tau_2(\varphi)(x) \geqslant \inf_{x\in\Omega_\delta} \int_\Omega \hat{k}_2(x,y)\varphi(y)\,dy \geqslant \frac{\delta^3}{12} \int_\Omega \varphi(y)\big(1-y^2\big)\,dy$$

which is the hypothesis (2.6) and this ends the proof. □

We deduce from Proposition 1 that if h_0 also satisfies (2.7) then, part (ii) of Theorem 1 is valid for $\tau = \tau_2$.

4.3. Example 3

We consider here an elastic deformation τ_3 given with the help of a "regularized kernel", defined by

$$\tau_3(p) = \int_\Omega k_3(x,y)p(y)\,dy$$

where k_3 is given by

$$k_3(x,y) = \log\left(\frac{M}{|x-y|+\epsilon}\right) \tag{4.8}$$

with

$$\epsilon > 0, \quad M > \epsilon + diam(\Omega). \tag{4.9}$$

Since $k_3 \in C^\infty(\bar{\Omega}\times\bar{\Omega})$ and $k_3 > 0$ it is clear that all the hypotheses (2.3)–(2.6) and (2.8) are satisfied for the operator $\tau = \tau_3$ with weight $w(x) \equiv 1$.

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